ON THE BIRATIONAL ANABELIAN SECTION CONJECTURE

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Abstract. Assuming the finiteness of the Shafarevich-Tate group of elliptic curves over number fields we make several observations on the birational Grotendieck anabelian setion conjecture. We prove that the birational setion conjecture for curves over number fields can be reduced to the case of elliptic curves. In this case we prove that, as a consequence of a result of Stoll, a section of the exact sequence of the absolute Galois group of an elliptic curve over a number field arises from a rational point if and only if the induced section of the corresponding (geometrically abelianised) arithmetic fundamental group of the elliptic curve arises from a rational point. We also prove that given any curve over a number field, there exists a double covering of this curve for which the birational setion conjecture holds true.

Contents

§0. Introduction. Let k be a field of characteristic 0, and X a proper, smooth, and geometrically connected (not necessarily hyperbolic) *algebraic curve* over k. Let K_X be the function field of X, K_X^{sep} a separable closure of K_X, and $\bar{k}$ the algebraic closure of k in K_X^{sep}. Write

$$G_X \overset{\text{def}}{=} \operatorname{Gal}(K_X^{\text{sep}}/K_X),$$

and

$$G_{\overline{X}} \overset{\text{def}}{=} \operatorname{Gal}(K_X^{\text{sep}}/K_{\overline{X}}),$$

where $K_{\overline{X}} \overset{\text{def}}{=} K_X.\overline{k}$ is the function field of the geometric fibre $\overline{X} \overset{\text{def}}{=} X \times_k \overline{k}$ of X. There exists a canonical exact sequence of profinite absolute Galois groups

$$1 \to G_{\overline{X}} \to G_X \xrightarrow{\text{pr}_X} G_k \to 1,$$

where $G_k = \operatorname{Gal}(\bar{k}/k)$. Let $x \in X(k)$ be a *rational* point. Then x determines a *decomposition subgroup* $D_x \subset G_X$, which is only defined up to conjugation by the

elements of $G_{\overline{X}}$, and which maps surjectively onto G_k via the natural projection $\mathrm{pr}_X : G_X \twoheadrightarrow G_k$. More precisely, D_x sits naturally in the following exact sequence

$$1 \to \hat{\mathbb{Z}}(1) \to D_x \to G_k \to 1.$$

The above exact sequence is known to be split. A *section* $G_k \to D_x$ of the natural projection $D_x \twoheadrightarrow G_k$ (i.e. a splitting of the above exact sequence) determines naturally a *section* $G_k \to G_X$ of the natural projection $\mathrm{pr}_X : G_X \twoheadrightarrow G_k$, whose image is contained in D_x. The *birational* version of the anabelian Grothendieck section conjecture for curves predicts that a *section*, or *splitting*, of the exact sequence

$$1 \to G_{\overline{X}} \to G_X \xrightarrow{\mathrm{pr}_X} G_k \to 1,$$

over a *finitely generated* field k of characteristic zero, necessarily arises from a *rational point* $x \in X(k)$ of the curve X as explained above. (cf. [Koenigsmann], and §1, for more details). More generally, one says that a field k possesses the *birational section property for curves* if a similar statement as above holds for any curve over k (cf. loc. cit.). A major breakthrough towards the birational section conjecture is the fundamental result of Koenigsmann, that p-adic local fields (i.e. finite extensions of $\mathbb{Q}_p$) possess the birational section property for curves (cf. [Koenigsmann]). Also, it is well-known that the field $\mathbb{R}$ of real numbers has the birational section property.

In this paper we make several observations, and prove several facts, regarding this conjecture. First, we prove that in order to verify that a field k has the birational section property it suffices to reduce to the case where $X = \mathbb{P}^1_k$ is the projective line (cf. Lemma 2.1), and more generally to the case of curves with a given genus $g \geq 1$ (cf. Proposition 2.2 and Corollary 2.3).

Let k be a number field. For each place v of k let k_v be the completion of k at v and $X_v \overset{\mathrm{def}}{=} X \times_k k_v$. Let $s : G_k \to G_X$ be a section of the natural projection $G_X \twoheadrightarrow G_k$. Then s gives *naturally* rise to sections $s_v : G_{k_v} \to G_{X_v}$ of the natural projection $G_{X_v} \twoheadrightarrow G_{k_v}$, for each place v of k (cf. Proof of Proposition 1.4). Each of these sections $s_v : G_{k_v} \to G_{X_v}$ arises from a rational point $x_v \in X(k_v)$, since p-adic local fields and the field of real numbers possess the birational section property for curves. We prove the following. Suppose there exists a rational point $x \in X(k)$ such that $x = x_v$ for each place v of k, i.e. the local sections s_v arise form a global rational point $x \in X(k)$, then the section $s : G_k \to G_X$ arises from the rational point x (cf. Proposition 5.3, and the Proof of Proposition 3.3 where a more precise statement is proved in the case where $X = \mathbb{P}^1_k$ is the projective line).

Assuming the *finiteness* of the Shafarevich-Tate groups of elliptic curves, we prove that the birational section conjecture for curves over number fields can be reduced to the case of *elliptic curves* (cf. Proposition 4.2). In the case of an *elliptic curve* E over a *number field* k, with *finite Shafarevich-Tate group*, a section $s : G_k \to G_E$ of the exact sequence

$$1 \to G_{\overline{E}} \to G_E \xrightarrow{\mathrm{pr}_E} G_k \to 1$$

of the absolute Galois group of the function field of E gives rise naturally to a section $\tilde{s} : G_k \to \Pi_E$ of the exact sequence

$$1 \to T\overline{E} \to \Pi_E \xrightarrow{\mathrm{pr}_E} G_k \to 1$$

of the arithmetic fundamental group Π_E of E, where $T\overline{E}$ is the Tate module of E. Fix a base point of the torsor of splittings of the exact sequence $1 \to T\overline{E} \to \Pi_E \xrightarrow{\mathrm{pr}_E} G_k \to 1$ which arises from the origin of E. then the *conjugacy class* of the section $s : G_k \to \Pi_E$ corresponds to an element of $H^1(G_k, T\overline{E})$ which we denote also $\tilde{s}$. We observe that $\tilde{s}$ lies in the subgroup (via Kummer theory) $E(k)^\wedge$ of $H^1(G_k, T\overline{E})$, where $E(k)^\wedge$ denotes the profinite completion of the group of rational points $E(k)$ (cf. Lemma 4.4). Furthermore, we prove that, as a consequence of a result of Stoll, the above birational section $s : G_k \to G_E$ arises from a rational point of E if and only the above element $\tilde{s} \in E(k)^\wedge$ lies in the discrete subgroup $E(k) \subset E(k)^\wedge$ (cf. Propsoition 4.6). More precisely, one can in the framework of the birational anabelian section conjecture give a (group-theoretic) characterisation of the *discrete* group $E(k)$ inside its *profinite* completion $E(k)^\wedge$ (cf. Proposition 4.7). Similar observations are made for birational sections in the case of curves of genus at least 2 (cf. Propsoition 5.2). Finally, we prove that given a proper, smooth, and geometrically connected curve X over a number field k there exists a *double covering* $X' \to X$ defined over k such that the birational section conjecture holds true for X', under the assumption that the Shafarevich-Tate groups of elliptic curves over k are finite, (cf. Lemma 5.5, and Remark 5.6).

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§1. The Birational Grothendieck Anabelian Section Conjecture. In this section we briefly recall, and explain, the content of the birational anabelian section conjecture of Grothendieck for curves (cf. [Grothendieck]). We also fix notations that will be used throughout this paper.

Let k be a field of characteristic 0, and X a proper, smooth, and geometrically connected (not necessarily hyperbolic) *algebraic curve* over k. Let K_X be the function field of X, K_X^{sep} a separable closure of K_X, and $\bar{k}$ the algebraic closure of k in K_X^{sep}. Write

$$G_X \overset{\mathrm{def}}{=} \mathrm{Gal}(K_X^{\mathrm{sep}}/K_X),$$

and

$$G_{\overline{X}} \overset{\mathrm{def}}{=} \mathrm{Gal}(K_X^{\mathrm{sep}}/K_{\overline{X}}),$$

where $K_{\overline{X}} \overset{\mathrm{def}}{=} K_X.\bar{k}$ is the function field of the geometric fibre $\overline{X} \overset{\mathrm{def}}{=} X \times_k \bar{k}$ of X. There exists a canonical exact sequence of profinite absolute Galois groups

$$1 \to G_{\overline{X}} \to G_X \xrightarrow{\mathrm{pr}_X} G_k \to 1, \tag{1}$$

where $G_k = \mathrm{Gal}(\bar{k}/k)$.

By a group-theoretic *section*, or a *splitting*, of the exact sequence (1) we mean a continuous homomorphism $s : G_k \to G_X$ such that $\mathrm{pr}_X \circ s = \mathrm{id}_{G_k}$. Let $x \in X(k)$ be a *rational* point of X. Then x determines a *decomposition subgroup* $D_x \subset G_X$, which is only defined up to conjugation by the elements of $G_{\overline{X}}$, and which maps

surjectively onto G_k via the natural projection $\mathrm{pr}_X : G_X \twoheadrightarrow G_k$. More precisely, D_x sits naturally in the following exact sequence

$$1 \to \hat{\mathbb{Z}}(1) \to D_x \to G_k \to 1. \tag{2}$$

The exact sequence (2) is known to be split. Indeed, the extension defined by extracting n-th roots, for all positive integers n, of a given local parameter at x defines a splitting of this sequence. The set of all splittings of the exact sequence (2) is a torsor under the Galois cohomology group $H^1(G_k, \hat{\mathbb{Z}}(1))$. A section $G_k \to D_x$ of the natural projection $D_x \twoheadrightarrow G_k$ (i.e. a splitting of the exact sequence (2)) determines naturally a section $G_k \to G_X$ of the natural projection $\mathrm{pr}_X : G_X \twoheadrightarrow G_k$, whose image is contained in D_x.

The Birational Grothendieck Anabelian Section Conjecture (BGASC) (cf. [Koenigsmann]). Assume that k is *finitely generated* over the prime field $\mathbb{Q}$. Let $s : G_k \to G_X$ be a group-theoretic *section* of the natural projection $\mathrm{pr}_X : G_X \twoheadrightarrow G_k$. Then the image $s(G_k)$ is contained in a decomposition subgroup $D_x \subset G_X$ associated to a *unique rational point* $x \in X(k)$. In particular, the existence of the section s implies that $X(k) \neq \varnothing$.

Definition 1.1. Let k be a field. We say that the BGASC *holds true* over k if for every proper, smooth, and geometrically connected algebraic curve X over k, and every group-theoretic *section* $s : G_k \to G_X$ of the natural projection $\mathrm{pr}_X : G_X \twoheadrightarrow G_k$, the image $s(G_k)$ is contained in a decomposition subgroup $D_x \subset G_X$ associated to a *unique rational point* $x \in X(k)$. In this case we say that the BGASC *holds true for the k-curve X.*

In connection with the BGASC, in the case where k is a *number field*, it is natural to formulate a p-adic version of this conjecture over p-adic local fields.

A p-adic Version of the Birational Grothendieck Anabelian Section Conjecture (p-adic BGASC) (cf. loc. cit.). Let $p > 0$ be a prime integer, and assume that k is a *finite extension* of $\mathbb{Q}_p$. Then the BGASC holds true over k. More precisely, let $s : G_k \to G_X$ be a group-theoretic *section* of the natural projection $G_X \twoheadrightarrow G_k$. Then the image $s(G_k)$ is contained in a decomposition subgroup $D_x \subset G_X$ associated to a *unique rational point* $x \in X(k)$. In particular, the existence of the section s implies that $X(k) \neq \varnothing$.

Remark 1.2. The uniqueness of the rational point $x \in X(k)$ mentioned in the BGASC, and its p-adic variant, is well-known if such a point exists. Indeed, any conjugates of two decomposition subgroups of G_X corresponding to distinct closed points of X have trivial intersection. Thus, in order to establish these conjectures, it suffices to establish the existence of a rational point x such that a corresponding decomposition group D_x contains the image of the section s.

A major breakthrough towards the BGASC is the following fundamental result concerning the p-adic BGASC, and which is du to Koenigsmann (cf. [Koenigsmann]).

Theorem 1.3 (Koenigsmann). *The p-adic version of the BGASC holds true. More precisely, assume that k is a finite extension of $\mathbb{Q}_p$. Let $s : G_k \to G_X$ be a group-theoretic section of the natural projection $G_X \twoheadrightarrow G_k$. Then the image $s(G_k)$*

is contained in a decomposition subgroup D_x associated to a unique rational point $x \in X(k)$. In particular, the existence of the section s implies that $X(k) \neq \varnothing$.

This result has been strengthened by Pop, who proved a *$\mathbb{Z}/p\mathbb{Z}$-meta-abelian* version of this theorem (see [Pop] for more details). An important consequence of Theorem 1.3 is the following, which was already observed in [Koenigsmann].

Proposition 1.4. *Let k be a number field and X a proper, smooth, and geometrically connected (not necessarily hyperbolic) curve over k. Assume that there exists a section $s : G_k \to G_X$ of the natural projection $G_X \twoheadrightarrow G_k$. Then the section s gives rise to an adelic point $(x_v)_v \in X(\mathbb{A}_k)$. Moreover, x_v (resp. the connected component containing x_v) is uniquely determined by the section s in the case where v is a finite place (resp. v is a real place). Here, $\mathbb{A}_k$ denotes the ring of adèles of k, and v runs over all places of k.*

Proof. See [Koenigsmann], Corollary 2.6. In fact one can prove a more precise statement than in [Koenigsmann] (cf. loc. cit.). For each place v of k, let k_v^h (resp. k_v) be the henselisation of k at v (resp. the completion of k at v), and $X_v^h \overset{\text{def}}{=} X \times_k k_v^h$ (resp. $X_v \overset{\text{def}}{=} X \times_k k_v$). The section s induces naturally a section $s_v^h : G_{k_v} \to G_{X_v^h}$ of the natural projection $G_{X_v^h} \twoheadrightarrow G_{k_v}$ (here, we fix an identification of $G_{k_v} \overset{\sim}{\to} G_{k_v}^h$ with a decomposition subgroup of G_k at the place v). By a (an unpublished) result of Tamagawa, the section s_v^h can be lifted to a section $s_v : G_{k_v} \to G_{X_v}$ of the natural projection $G_{X_v} \twoheadrightarrow G_{k_v}$ (cf. [Saïdi], Theorem 5.6). More precisely, one can construct a section $s_v : G_{k_v} \to G_{X_v}$ which fits into the following commutative diagram

$$\begin{tikzcd}
G_{k_v} \arrow[r, "s_v"] \arrow[d, "\mathrm{id}"'] & G_{X_v} \arrow[d] \\
G_{k_v} \arrow[r, "s_v^h"] \arrow[d] & G_{X_v^h} \arrow[d] \\
G_k \arrow[r, "s"] & G_X
\end{tikzcd}$$

where the right top vertical map is a natural surjection, and the left low vertical map is an embeeding. By Theorem 1.3 above of Koenigsmann, for a finite place v, the image $s_v(G_{k_v})$ is contained in a decomposition subgroup D_{x_v} associated to a unique rational point $x_v \in X(k_v)$. This is also true for the archimedian places (the so-called real section conjecture holds true). Note that in the case where v is a real place only the connected component of $X(k_v)$ containing x_v is well determined by the section s. Thus, to the section s is associated *naturally* an adelic point $(x_v)_v \in X(\mathbb{A}_k)$ with the required properties. $\square$

§**2. Reduction of the Birational Section Conjecture to Curves with a given Genus.** In this section we state and prove our main observation concerning the BGASC, that it can be reduced to curves with a given genus. Our first observation is that the BGASC can be (easily) reduced to the case of the *projective line*.

Lemma 2.1. *Let k be a field. Assume that the BGASC holds true for $\mathbb{P}_k^1$ (cf. Definition 1.1). Then the BGASC holds true for any k-curve X which is projective, smooth, and geometrically connected.*

Proof. Let X be a projective, smooth, and geometrically connected algebraic curve over k. Let $f : X \to \mathbb{P}^1_k$ be a finite morphism, which corresponds to a finite field extension $K_X/k(T)$ where $k(T) \overset{\text{def}}{=} K_{\mathbb{P}^1_k}$. We have a natural commutative diagram of exact sequences of profinite Galois groups

$$\begin{tikzcd}
1 \arrow[r] & G_{\overline{X}} \arrow[r] \arrow[d] & G_X \arrow[r, "\mathrm{pr}_X"] \arrow[d] & G_k \arrow[r] \arrow[d, "\mathrm{id}"'] & 1 \\
1 \arrow[r] & G_{\mathbb{P}^1_{\overline{k}}} \arrow[r] & G_{\mathbb{P}^1_k} \arrow[r, "\mathrm{pr}_{\mathbb{P}^1_k}"] & G_k \arrow[r] & 1
\end{tikzcd}$$

where the left and middle vertical maps are natural inclusions. Here, $G_{\mathbb{P}^1_k} \overset{\text{def}}{=} \mathrm{Gal}(K_X^{\text{sep}}/k(T))$, and $G_{\mathbb{P}^1_{\overline{k}}} \overset{\text{def}}{=} \mathrm{Gal}(K_X^{\text{sep}}/\overline{k}(T))$. Let $s : G_k \to G_X$ be a group-theoretic section of the natural projection $\mathrm{pr}_X : G_X \twoheadrightarrow G_k$. The image of $s(G_k)$ in $G_{\mathbb{P}^1_k}$, via the natural embedding $G_X \hookrightarrow G_{\mathbb{P}^1_k}$, determines a group-theoretic section $\tilde{s} : G_k \to G_{\mathbb{P}^1_k}$ of the natural projection $G_{\mathbb{P}^1_k} \twoheadrightarrow G_k$. Assume that the BGASC holds true for $\mathbb{P}^1_k$. Then the image $\tilde{s}(G_k)$ of the section $\tilde{s}$ is contained in a decomposition subgroup $D_y \subset G_{\mathbb{P}^1_k}$ associated to a unique rational point $y \in \mathbb{P}^1(k)$. The intersection $D_x \overset{\text{def}}{=} D_y \cap G_X$ is then the decomposition group associated to a unique point $x \in X$, which is necessarily k-rational since D_x maps surjectively onto G_k via the natural projection $G_X \twoheadrightarrow G_k$. Moreover, we have $s(G_k) \subseteq D_x$. $\square$

Next, we prove that the BGASC over any field can be reduced to the case of curves with a given genus $g \geq 1$. We refer to the discussion in §3 for the case of genus 0 curves over number fieds.

Proposition 2.2. *Let k be a field, and $g \geq 1$ an integer. Assume that the BGASC holds true for genus g proper, smooth, and geometrically connected curves over k. Then the BGASC holds true for the projective line over k.*

In particular, as a consequence of Lemma 2.1, and Proposition 2.2, one deduces immediately the following.

Corollary 2.3. *Let k be a finitely generated field over $\mathbb{Q}$, and $g \geq 1$ an integer. Assume that the BGASC holds true for genus g proper, smooth, and geometrically connected curves over k. Then the BGASC holds true for any projective, smooth, and geometrically connected curve X over k.*

Proof of Proposition 2.2. Recall the exact sequence of absolute Galois groups

$$1 \to G_{\mathbb{P}^1_{\overline{k}}} \to G_{\mathbb{P}^1_k} \xrightarrow{\mathrm{pr}_{\mathbb{P}^1_k}} G_k \to 1.$$

Let $s : G_k \to G_{\mathbb{P}^1_k}$ be a section of the natural projection $G_{\mathbb{P}^1_k} \twoheadrightarrow G_k$. Let $\overline{\Delta}$ be an open subgroup of $G_{\mathbb{P}^1_{\overline{k}}}$ corresponding to a finite morphism $\tilde{f} : \tilde{X} \to \mathbb{P}^1_{\overline{k}}$, where $\tilde{X}$ is a genus g proper, smooth, and connected curve over $\overline{k}$. Assume moreover that the finite morphism $\tilde{f} : \tilde{X} \to \mathbb{P}^1_{\overline{k}}$ is defined over k, in which case $\overline{\Delta}$ is stable under the natural action of $s(G_k)$ on $G_{\mathbb{P}^1_{\overline{k}}}$ via inner automorphisms. Write $\Delta \overset{\text{def}}{=} \overline{\Delta}.s(G_k)$. Then Δ is an open subgroup of $G_{\mathbb{P}^1_k}$ which corresponds to a finite morphism f :

$X \to \mathbb{P}^1_k$ where X is a projective, smooth, and geometrically connected k-curve. Let $G_X \overset{\mathrm{def}}{=} \mathrm{Gal}(K^{\mathrm{sep}}/K_X) = \Delta$, and $G_{\overline{X}} \overset{\mathrm{def}}{=} \mathrm{Gal}(K^{\mathrm{sep}}/K_{\overline{X}}) = \overline{\Delta}$, where $\overline{X} \overset{\mathrm{def}}{=} X \times_k \overline{k}$, and $K^{\mathrm{sep}} \overset{\mathrm{def}}{=} K^{\mathrm{sep}}_{\mathbb{P}^1_k}$. We have a natural commutative diagram of exact sequences of absolute Galois groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G_{\overline{X}} = \overline{\Delta} & \longrightarrow & G_X = \Delta & \xrightarrow{\mathrm{pr}_X} & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
1 & \longrightarrow & G_{\mathbb{P}^1_{\overline{k}}} & \longrightarrow & G_{\mathbb{P}^1_k} & \xrightarrow{\mathrm{pr}_{\mathbb{P}^1_k}} & G_k & \longrightarrow & 1
\end{array}$$

Note that by construction we have a natural isomorphism $\tilde{X} \overset{\sim}{\to} \overline{X}$ over $\overline{k}$. In particular, X is a genus g curve. Also, by construction, the group-theoretic section $s : G_k \to G_{\mathbb{P}^1_k}$ naturally restricts to a group-theoretic section $s : G_k \to G_X$ of the natural projection $G_X \twoheadrightarrow G_k$. Furthermore, in order to show that $s(G_k)$ is contained in a decomposition subgroup D_y associated to a rational point $y \in \mathbb{P}^1(k)$, it suffices to show that $s(G_k)$ is contained in a decomposition subgroup $D_x \subset G_X$ associated to a rational point $x \in X(k)$. Indeed, if $s(G_k) \subseteq D_x$, where $x \in X(k)$, then $s(G_k) \subseteq D_y$ is contained in a decomposition subgroup associated to the rational point $y \in \mathbb{P}^1(k)$ which is the image of x under the morphism $f : X \to \mathbb{P}^1_k$ corresponding to the inclusion $G_X \subset G_{\mathbb{P}^1_k}$. Moreover, $s(G_k) \subset D_x$ for a unique rational point $x \in X(k)$ if we assume that the BGASC holds true for X. $\square$

§3. Birational Sections for Genus 0 Curves over Number Fields. In this section we discuss the BGASC in the case of genus 0 curves, mainly over number fields. We first observe the following.

Proposition 3.1. *Let k be a number field and X a proper, smooth, and geometrically connected genus 0 curve over k. Assume that there exists a section $s : G_k \to G_X$ of the natural projection $G_X \twoheadrightarrow G_k$. Then $X(k) \neq \varnothing$. In particular, $X \overset{\sim}{\to} \mathbb{P}^1_k$ is a projective line.*

Proof. Indeed, the set of adelic point $X(\mathbb{A}_k) \neq \varnothing$ is non-empty by Proposition 1.4. Hence the set of rational points $X(k) \neq \varnothing$ is non empty, since the Hasse principle for rational points holds for X. $\square$

3.2. Let k be a number field, and $X = \mathbb{P}^1_k$ the projective line over k. Let $\infty \in X(k) = \mathbb{P}^1_k(k)$ be a rational point. Let $s : G_k \to G_X$ be a section of the natural projection $G_X \twoheadrightarrow G_k$. For each place v of k, let k_v be the completion of k at v, and $X_v \overset{\mathrm{def}}{=} X \times_k k_v$. The section s induces naturally a section $s_v : G_{k_v} \to G_{X_v}$ of the natural projection $G_{X_v} \twoheadrightarrow G_{k_v}$ (here, we fix an identification of G_{k_v} with a decomposition subgroup of G_k at the place v). More precisely, there exists a section s_v which fits into the following commutative diagram

$$\begin{array}{ccc}
G_{k_v} & \xrightarrow{s_v} & G_{X_v} \\
\downarrow & & \downarrow \\
G_k & \xrightarrow{s_v^h} & G_X
\end{array}$$

where the right vertical map is the natural one (cf. Proof of Proposition 1.4). We know that the section s_v arises from a unique rational point $x_v \in X(k_v)$ (cf.

Theorem 1.3). If the section s arises from a rational point $x \in X(k)$. Then, after observing the natural action of $PGL_2(k)$ on G_X, we can assume that $x = \infty$. The section s_v would then also arise from the point ∞. Reciprocally, We can prove the following.

Proposition 3.3. *We use the same notations and assumptions as in 3.2. Assume that for each place v of k we have $x_v = \infty$. In other words assume the image $s_v(G_{k_v}) \subset D_{\infty,v}$ is contained in a decomposition group $D_{\infty,v} \subset G_{X_v}$ associated to the point $\infty \in X(k) \subset X(k_v)$. Then the section s arises from the rational point ∞, i.e. the image $s(G_k) \subset D_\infty$ is contained in a decomposition group $D_\infty \subset G_k$ associated to ∞.*

Proof. One can write down a proof similar to the proof of Proposition 4.6, which resorts directly to Theorem 1.3 and a result of Stoll (cf. the Proof of Proposition 4.6). We will however prove a slightly more precise statement without resorting to the result of Stoll. We will show that there exists a *neighbourhood* of the section s, that is the absolute Galois group of the function field of an elliptic curve for which the BGASC holds (under the assumption of finiteness of Shafarevich-Tate groups of elliptic curves over k).

Let E be an elliptic curve over k with trivial Mordell-Weil rank (such curves exist, cf. Remark 5.6). Then E can be realised as a Galois cover $f : E \to \mathbb{P}^1_k$ of degree 2 of the projective line ramified above ∞ (in particular, the rational point ∞ lifts to a unique rational point $x \in E(k)$), and the absolute Galois group G_E naturally embeds $\iota : G_E \hookrightarrow G_{\mathbb{P}^1_k}$ in $G_{\mathbb{P}^1_k}$ as a normal subgroup of index 2. We have a natural commutative diagram

$$\begin{array}{ccc} G_E & \longrightarrow & G_k \\ {\scriptstyle\iota}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{id}} \\ G_{\mathbb{P}^1_k} & \longrightarrow & G_k \end{array}$$

We will show that G_E necessarily contains the image $s(G_k)$ of the section s. For each place v of k we have a natural commutative diagram

$$\begin{array}{ccc} G_{E_v} & \longrightarrow & G_{k_v} \\ {\scriptstyle\iota_v}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{id}} \\ G_{\mathbb{P}^1_{k_v}} & \longrightarrow & G_{k_v} \end{array}$$

where the left vertical embedding is naturally induced by ι. Moreover, the image $s_v(G_{k_v})$ of the section s_v, and all its conjugate, are contained in G_{E_v} (since the point ∞ lifts to a unique rational point of E_v, and G_{E_v} is a normal subgroup of $G_{\mathbb{P}^1_{k_v}}$). On the other hand, G_k is normally topologically generated by the decomposition subgroups G_{k_v} as follows from the Chebotarev density theorem. From this follows that G_E is normally topologically generated by the images of the G_{E_v}, where v runs over all places of k. Hence, G_E contains $s(G_{k_v})$, and all its conjugates, for all places v. Thus, G_E contains $s(G_k)$, and the section s naturally restricts to a section $s : G_k \to G_E$ of the natural projection $G_E \twoheadrightarrow G_k$, which arises from a rational point $y \in E(k)$ by Corollary 4.8 (here we assume that the Shafarevich-Tate group of E is finite). In particular, the section $s : G_k \to G_{\mathbb{P}^1_k}$ arises from the rational point $x \in \mathbb{P}^1_k(k)$ which is the image of y under the above morphism $f : E \to \mathbb{P}^1_k$. $\square$

Remark/Question 3.4. We use the same notations as in 3.2. One can, after observing the action of $PGL_2(k)$, assume that for every finite set of places S of k one has $x_v = \infty$, since $PGL_2(k)$ is dense in $PGL_2(\mathbb{A}_k)$. Is it possible to prove that this leads to the same conclusion as in Proposition 3.3? If yes, this would prove the BGASC for $\mathbb{P}^1_k$ in the case where k is a number field.

§**4. Birational Sections for Genus** 1 **Curves over Number Fields.** In this section we discuss the BGASC for genus 1 curves, mainly over number fields. First, we observe that the existence of a birational section for a genus 1 curve over a number field implies, assuming the finiteness of the Shafarevich-Tate groups for elliptic curves, that this curve is an elliptic curve. More precisley, we have the following.

Proposition 4.1. *Let k be a number field and X a proper, smooth, geometrically connected genus* 1 *curve over k. Assume that the Shafarevich-Tate groups of elliptic curves over k are finite. Assume there exists a section $s : G_k \to G_X$ of the natural projection $G_X \twoheadrightarrow G_k$. Then $X(k) \neq 0$. In particular, X is an elliptic curve.*

An immediate consequence of Proposition 4.1, and Proposition 2.2, is that the BGASC for *curves over number fields* can be reduced to the case of *elliptic curves over number fields* (assuming the finiteness of the Shafarevich-Tate groups for elliptic curves). More precisely, we have the following.

Proposition 4.2. *Let k be a number field. Assume that the Shafarevich-Tate groups of elliptic curves over k are finite, and that the BGASC holds true for all elliptic curves over k. Then the BGASC holds true for any projective, smooth, and geometrically connected curve X over k.*

Proof of Proposition 4.1. Recall the exact sequence of absolute Galois groups

$$1 \to G_{\overline{X}} \to G_X \xrightarrow{\mathrm{pr}_X} G_k \to 1.$$

Let $s : G_k \to G_X$ be a section of the natural projection $G_X \twoheadrightarrow G_k$. By assumption, X is a genus 1 curve. Moreover, X is a principal homogeneous space over k under its jacobian E' which is an elliptic curve over k, and corresponds to an element of the Galois cohomology group $H^1(G_k, E')$. Next, assuming that the Shafarevich-Tate group of E' is finite, we will show that $X \xrightarrow{\sim} E'$ is an elliptic curve. The existence of the section $s : G_k \to G_X$ implies that $X(\mathbb{A}_k) \neq \varnothing$ (cf. Proposition 1.4). Thus, to the section s is associated an adelic point $(x_v)_v \in X(\mathbb{A}_k)$ (cf. loc. cit.). The adelic point $(x_v)_v \in X(\mathbb{A}_k)$ survives every finite étale abelian descent obstruction (cf. [Stoll], Definition 5.2), as follows easily from the existence of the global section s (see also [Harari-Stix], Proposition 1.1), i.e. $(x_v)_v \in X(\mathbb{A}_k)^{\mathrm{f-ab}}$ in the terminology of Stoll, where $X(\mathbb{A}_k)^{\mathrm{f-ab}}$ is the set of adelic points cut out by the finite étale abelian descent conditions (cf. loc. cit. Definition 5.4). For a different argument to deduce the existence of a point in $X(\mathbb{A}_k)^{\mathrm{f-ab}}$ one may also use similar arguments as in the proof of Theorem 3.2 in [Harari-Stix]. On the other hand one has the following equality $X(\mathbb{A}_k)^{\mathrm{f-ab}} = X(\mathbb{A}_k)^{\mathrm{Br}}$ where $X(\mathbb{A}_k)^{\mathrm{Br}}$ denotes the set of adelic points cut out by the Brauer-Manin conditions, i.e. the Brauer-Manin set (cf. [Stoll], Corollary 7.3). The non-emptiness of the Brauer set $X(\mathbb{A}_k)^{\mathrm{Br}}$ implies, under the assumption that the Shafarevich-Tate group of E' is finite, that $X(k) \neq \varnothing$ by a result of Manin (cf. [Manin]), hence $X \xrightarrow{\sim} E'$ is an elliptic curve. $\square$

4.3. Next, we will discuss the BGASC in the case of an elliptic curve over a number field. In what follows we will assume that k is a **number field**, and E is an **elliptic curve over k with finite Shafarevich-Tate group**.

Recall the exact sequence of absolute Galois groups

$$1 \to G_{\overline{E}} \to G_E \xrightarrow{\mathrm{pr}_E} G_k \to 1.$$

Let Π_E be the quotient of G_E which corresponds to the maximal *everywhere unramified* extension of K_E contained in K_E^{sep}. Thus, Π_E is the *arithmetic étale fundamental group* of E. We have a natural commutative diagram of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & G_{\overline{E}} \arrow[r] \arrow[d] & G_E \arrow[r, "\mathrm{pr}_E"] \arrow[d] & G_k \arrow[r] \arrow[d, "\mathrm{id}"'] & 1 \\
1 \arrow[r] & \Pi_{\overline{E}} \arrow[r] & \Pi_E \arrow[r, "\mathrm{pr}_E"] & G_k \arrow[r] & 1
\end{tikzcd}$$

where $\Pi_{\overline{E}}$ is the étale fundamental group of $\overline{E} \overset{\mathrm{def}}{=} E \times_k \bar{k}$, which is naturally identified with the *Tate module* $T\overline{E}$ of $\overline{E}$. The left and middle vertical maps in the above diagram are surjective. We fix a *base point* of the torsor of splittings of the exact sequence $1 \to \Pi_{\overline{E}} \to \Pi_E \xrightarrow{\mathrm{pr}_E} G_k \to 1$, which corresponds to the splitting arising from the origin of the elliptic curve E. The set of splittings of the above sequence is then a torsor under the Galois cohomology group $H^1(G_k, \Pi_{\overline{E}})$. Let $s : G_k \to G_E$ be a group-theoretic *section* of the natural projection $\mathrm{pr}_E : G_E \twoheadrightarrow G_k$. Then s induces naturally a group-theoretic section $\tilde{s} : G_k \to \Pi_E$ of the natural projection $\mathrm{pr}_E : \Pi_E \twoheadrightarrow G_k$. We have a commutative diagram

$$\begin{tikzcd}
G_k \arrow[r, "s"] \arrow[d, "\mathrm{id}"'] & G_E \arrow[d] \\
G_k \arrow[r, "\tilde{s}"] & \Pi_E
\end{tikzcd}$$

where the right vertical map is the natural surjection. The conjugacy class of the section $\tilde{s}$ corresponds to a unique element of $H^1(G_k, \Pi_{\overline{E}})$, which we will denote also $\tilde{s} \in H^1(G_k, \Pi_{\overline{E}})$. We have a natural exact sequence arising from Kummer theory

$$1 \to E(k)^\wedge \to H^1(G_k, \Pi_{\overline{E}}) \to TH^1(G_k, E) \to 1, \tag{3}$$

where $E(k)^\wedge \overset{\mathrm{def}}{=} \varprojlim_{n \geq 1} \frac{E(k)}{nE(k)}$ is the profinite completion of the finitely generated discrete group $E(k)$, $H^1(G_k, \Pi_{\overline{E}})$ is the profinite Galois cohomology group of the continuous G_k-module $\Pi_{\overline{\Pi}}$, and $TH^1(G_k, E)$ is the Tate module of the Galois cohomology group $H^1(G_k, E(\overline{k}))$.

Lemma 4.4. *We use the same notations and assumptions as in 4.3. The element $\tilde{s} \in H^1(G_k, \Pi_{\overline{E}})$, corresponding to the section $\tilde{s} : G_k \to \Pi_E$, lies in the subgroup $E(k)^\wedge \subset H^1(G_k, \Pi_{\overline{E}})$.*

Proof. The existence of the section s gives rise *naturally* to an adelic point $(x_v)_v \in E(\mathbb{A}_k)$ (cf. Proof of Proposition 1.4), where x_v is uniquely determined at the finite

places v. At a (possible) real place v of k only the connected component of $E(k_v)$ containing x_v is well defined. We have a natural commutative diagram of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & E(k)^\wedge \arrow[r] \arrow[d] & H^1(G_k, \Pi_{\overline{E}}) \arrow[r] \arrow[d] & TH^1(G_k, E) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \prod_v E(k_v)^\wedge \arrow[r] & \prod_v H^1(G_{k_v}, \Pi_{\overline{E}}) \arrow[r] & \prod_v TH^1(G_{k_v}, E) \arrow[r] & 1
\end{tikzcd}$$

where the product in the lower exact sequence is taken over all places v of k. The image of $\tilde{s}$ in $\prod_v H^1(G_{k_v}, \Pi_{\overline{E}})$ is $(x_v)_v \in \prod_v E(k_v)^\wedge$. The kernel of the natural map $TH^1(G_k, E) \to \prod_v TH^1(G_k, E)$ is the Tate module of the Shafarevich-Tate group of E; it is trivial if we assume that the latter is finite. Hence the image of $\tilde{s}$ in $TH^1(G_k, E)$ is trivial and $\tilde{s}$ lies in $E(k)^\wedge$ as claimed. Note that in the above commutative diagram it is well-known that the left and middle vertical maps are injective.

Alternatively, the adelic point $(x_v)_v \in E(\mathbb{A}_k)$ survives every finite étale abelian descent obstruction (cf. [Stoll], Definition 5.2), as follows easily from the existence of the global section s (see also [Harari-Stix], Proposition 1.1), i.e. $(x_v)_v \in E(\mathbb{A}_k)^{\mathrm{f-ab}}$ in the terminology of Stoll, where $E(\mathbb{A}_k)^{\mathrm{f-ab}}$ is the set of adelic points cut out by the finite étale abelian conditions (cf. loc. cit. Definition 5.4). This implies that $\tilde{s}$ lies in the Selmer group $\mathrm{Sel}(k, E)^\wedge \subset H^1(G_k, \Pi_{\overline{E}})$ by a result of Stoll (cf. [Stoll], the discussion preceding Corollary 6.2). Furthermore, we have a natural identification $E(k)^\wedge \xrightarrow{\sim} \mathrm{Sel}(k, E)^\wedge$, since we assumed the Shafarevich-Tate group of E to be finite (cf. loc. cit.). Hence $\tilde{s} \in E(k)^\wedge$. $\square$

Remark 4.5. In fact, one can show slightly more than the statement in Lemma 4.4. For a finite closed subscheme $S \subset E$ denote by J_S the corresponding generalised jacobian with modulus S. Write $J_S(k)^\wedge$ for the profinite completion of the group of k-rational points $J_S(k)$ of J_S. We have a natural homomorphism $\varphi : \varprojlim_S J_S(k)^\wedge \to E(k)^\wedge$, where $\varprojlim_S J_S(k)^\wedge$ is the projective limit of the $J_S(k)^\wedge$'s. One can show that the element $\tilde{s}$ as in Lemma 4.4 lies in the image $\mathrm{Im}\,\varphi$ of the above homomorphism φ.

In the framework of the above discussion one can characterise, using a result of Stoll, those sections $s : G_k \to G_E$ which arise from rational points as follows.

Proposition 4.6. *We use the same notations and assumptions as in 4.3. The image of the section $s : G_k \to G_E$ is contained in the decomposition group D_x associated to a rational point $x \in E(k)$ if and only if the induced section $\tilde{s} : G_k \to \Pi_E$ corresponds to an element $\tilde{s} \in H^1(G_k, \Pi_{\overline{E}})$ which lies in the subgroup $E(k)$ of $E(k)^\wedge \subset H^1(G_k, \Pi_{\overline{E}})$. (We already know that the element s lies in $E(k)^\wedge$ by Lemma 4.4.)*

Proof. Note that the discrete group $E(k)$ naturally embeds into its profinite completion $E(k)^\wedge$, since it is finitely generated. First, one easily observes that if the section s arises from a rational point, i.e. if its image $s(G_k) \subset D_x$ is contained in a decomposition group associated to a rational point $x \in E(k)$, then the corresponding element $\tilde{s} \in H^1(G_k, \Pi_{\overline{E}})$ equals $s_x \in E(k)$, where $s_x \in H^1(G_k, \Pi_{\overline{E}})$ is the element corresponding to $x \in E(k)$. Here $E(k)$ is viewed as a subgroup

of $H^1(G_k, \Pi_{\overline{E}})$ via the Kummer sequence, the natural map $E(k) \to E(k)^\wedge$ being injective.

Second, assume that $\tilde{s} = s_x \in E(k)$ for some rational point (necessarily unique) $x \in E(k)$. We will show that the image $s(G_k) \subset G_E$ of the section s is contained in a decomposition group D_x associated to the rational point x. We use the following well-known argument in anabelian geometry. In order to show that $s(G_k) \subseteq D_x$ it suffices to show (using a limit argument, and Faltings theorem on the finiteness of the set of rational points of a smooth, hyperbolic, connected, and proper curve over a number field) that for every open subgroup H of G_E corresponding to a finite (possibly ramified) morphism $Y \to E$, where Y has genus at least 2, and such that $s(G_k) \subset H$, we have $Y(k) \neq \varnothing$. Indeed, in this case the projective limit $\varprojlim_Y Y(k)$, where the limit is taken over all such Y's, is non empty. Consider a pro-point in $\varprojlim_Y Y(k)$, and its image $x' \in E(k)$. Then $s(G_k) \subseteq D_{x'}$, where $D_{x'}$ is a decomposition subgroup associated to x'. Moreover, $x' = x$ necessarily.

Next, let $H \subseteq G_X$ be an open subgroup corresponding to a finite morphism $g : Y \to E$, where Y has genus at least 2, and such that $s(G_k) \subset H$. Then H is naturally identified with the absolute Galois group $G_Y \overset{\text{def}}{=} \mathrm{Gal}(K_E^{\mathrm{sep}}/K_Y)$, and the section s restricts to a section $s : G_k \to G_Y$ of the natural projection $G_Y \twoheadrightarrow G_k$. Similar arguments as the one used in the proof of Proposition 4.2 imply that the existence of the section $s : G_k \to G_Y$ gives rise to an adelic point $(y_v)_v \in Y(\mathbb{A}_k)$ which survives every finite étale abelian descent obstruction, i.e. $(y_v)_v \in Y(\mathbb{A}_k)^{\mathrm{f-ab}} \neq \varnothing$. Moreover, the image of $(y_v)_v$ in $E(\mathbb{A}_k)$ via the natural map $Y(\mathbb{A}_k) \to E(\mathbb{A}_k)$, which is induced by the natural morphism $g : Y \to E$, coincides with the adelic point $(x_v)_v \in E(\mathbb{A}_k)$ arising from the rational point $x \in E(k)$. Let $Z \subset Y$ be the preimage (as a subscheme) of the rational point $x \in E(k)$. Thus, Z is a finite k-scheme, and $(y_v)_v \in Z(\mathbb{A}_k)$. We have $Z(k) = Z(\mathbb{A}_k) \cap Y(\mathbb{A}_k)^{\mathrm{f-ab}}$ by a result of Stoll (cf. [Stoll], Theorem 8.2). In particular, $Z(k) \neq \varnothing$. Thus, $Y(k) \neq \varnothing$. This finishes the proof of Proposition 4.6. $\square$

In fact, the validity of the BGASC for elliptic curves over number fields gives a characterisation of the discrete group of rational points of an elliptic curve inside its profinite completion. More precisely, we have the following which follows easily from Proposition 4.6.

Proposition 4.7. *We use the same notations as above. Let E be an elliptic curve over a number field k. Assume that the BGASC holds true for E (cf. Definition 1.1). Let $\tilde{s} \in E(k)^\wedge$, which we view as an element of $H^1(G_k, \Pi_{\overline{E}})$. Then $\tilde{s}$ lies in $E(k)$ if and only if a corresponding section $\tilde{s} : G_k \to \Pi_E$ of the natural projection $\Pi_E \twoheadrightarrow G_k$ can be lifted to a section $s : G_k \to G_E$ of the natural projection $G_E \twoheadrightarrow G_k$, i.e. if there exists a section $s : G_k \to G_E$ and a commutative diagram*

$$\begin{array}{ccc} G_k & \xrightarrow{s} & G_E \\ {\scriptstyle\mathrm{id}}\downarrow & & \downarrow \\ G_k & \xrightarrow{\tilde{s}} & \Pi_E \end{array}$$

where the right vertical map is the natural surjection.

Proposition 4.7 implies immediately the following which was observed by Stoll (cf. [Stoll], Remark 8.9). See also [Harari-Stix] Theorem 3.5.

Corollary 4.8. *Let k be a number field, and E an elliptic curve over k. Assume that the Shafarevich-Tate group of E is finite, and that the rank of the Mordell-Weil group of E is trivial, i.e. $E(k)$ finite. Then the BGASC holds true for E.*

§5. Birational Sections for Genus $g \geq 2$ Curves over Number Fields.

In this section we will establish some observations on the BGASC in the case of genus $g \geq 2$ curves over number fields.

5.1. Assume that X is a proper, smooth, **hyperbolic**, and geometrically connected **curve** over a **number field** k. Assume that the **Shafarevich-Tate group** of the jacobian $J \overset{\text{def}}{=} J_X$ of X is **finite**. Let $s : G_k \to G_X$ be a section of the natural projection $G_X \twoheadrightarrow G_k$. Then X has a rational divisor of degree 1 (cf. [Esnault-Wittenberg]), and we can embed X into J. Let Π_J be the arithmetic fundamental group of J which sits naturally in an exact sequence

$$0 \to T\overline{J} \to \Pi_J \to G_k \to 1,$$

where $T\overline{J}$ is the Tate module of $\overline{J} \overset{\text{def}}{=} J \times_k \bar{k}$. Thus, Π_J corresponds naturally to the quotient of G_X which is the geometrically abelian étale fundamental group of X. We fix a base point of the torsor of splittings of the above exact sequence which arises from the splitting associated to the zero section. Recall the Kummer exact sequence

$$0 \to J(k)^\wedge \to H^1(G_k, T\overline{J}) \to TH^1(G_k, J) \to 1.$$

Similar arguments used in the proof of Proposition 4.6 yield the following.

Proposition 5.2. *We use the same notations and hypothesis as in 5.1. Let $s : G_k \to G_X$ be a section of the natural projection $G_X \twoheadrightarrow G_k$, $\tilde{s} : G_k \to \Pi_J$ the section of the natural projection $\Pi_J \twoheadrightarrow G_k$ which is naturally induced by s, and $\tilde{s} \in H^1(G_k, T\overline{J})$ the corresponding element of $H^1(G_k, T\overline{J})$. Then $\tilde{s}$ lies in the subgroup $J(k)^\wedge$ of $H^1(G_k, T\overline{J})$. Moreover, the image $s(G_k) \subset G_X$ of the section s is contained in the decomposition group D_x associated to a rational point $x \in X(k)$ if and only if the above elements $\tilde{s} \in J(k)^\wedge$ lies in the subgroup $J(k)$ of $J(k)^\wedge$.*

One can deduce, as a consequence of Proposition 5.2, the following.

Proposition 5.3. *We use the same notations and hypothesis as in 5.1. Let $s : G_k \to G_X$ be a section of the natural projection $G_X \twoheadrightarrow G_k$, and for each place v of k denote by $s_v : G_{k_v} \to G_{X_v}$ the corresponding section of the natural projection $G_{X_v} \twoheadrightarrow G_{k_v}$ (cf. proof of proposition 1.4). Let $x \in X(k)$ be a rational point. Assume that for each place v of k the section s_v arises from $x \in X(k) \subset X(k_v)$. In other words the image $s_v(G_{k_v}) \subset \tilde{D}_x$ is contained in a decomposition group $\tilde{D}_x \subset G_{X_v}$ associated to the point $x \in X(k_v)$. Then the section s arises from the rational point x, i.e. the image $s(G_k) \subset D_x$ is contained in a decomposition group $D_x \subset G_X$ associated to the rational point x.*

Proof. Indeed, with the same notations as in Proposition 5.2, in this case we have $\tilde{s} = x$ as an element of $X(k) \subset J(k) \subset J(k)^\wedge$. $\square$

Remark 5.4. The above discussion in the case of a curve X of genus at least 2 is related to the adelic-Mordell conjecture of Stoll (cf. [Stoll]), which predicts that inside $\prod_v J(k_v)$ the intersection $J(k)^\wedge \cap \prod_v X(k_v)$ is exactly $X(k)$. In fact

the validity of Stoll's conjecture would imply, with the notation in Proposition 5.2, that $\tilde{s}$ lies automatically in $J(k)$, hence the validity of the BGASC for X would follow. However, in the case of an elliptic curve, Proposition 4.6 does not seem to be a priori related to Stoll's conjecture and the results in [Stoll].

Finally, we observe the following.

Lemma 5.5. *Let k be a number field and X a proper, smooth, and geometrically connected curve over k. Assume that there exists an elliptic curve E over k with trivial Shafarevich-Tate group and with trivial Mordell-Weil rank. Then there exists a finite morphism $f : X' \to X$ of degree $\deg(f) \leq 2$ such that the BGASC holds true for X' as a k-curve.*

Proof. Let E be an elliptic curve over k with trivial Shafarevich-Tate group and with trivial Mordell-Weil rank. Let $\tilde{g} : X \to \mathbb{P}^1_k$ be a finite morphism, and $\tilde{f} : E \to \mathbb{P}^1_k$ a morphism of degree 2. Let $X' \overset{\text{def}}{=} X \times_{\mathbb{P}^1_k} E$. We have a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ X & \xrightarrow{\tilde{g}} & \mathbb{P}^1_k \end{array}$$

where $f : X' \to X$ is a finite morphism of degree $\deg(f) \leq 2$. We choose the function $\tilde{g}$ so that X' is geometrically connected. We have a commutative diagram of exact sequences of absolute Galois groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_{\overline{X'}} & \longrightarrow & G_{X'} & \xrightarrow{\mathrm{pr}_X} & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \\ 1 & \longrightarrow & G_{\overline{E}} & \longrightarrow & G_E & \xrightarrow{\mathrm{pr}_{\mathbb{P}^1_k}} & G_k & \longrightarrow & 1 \end{array}$$

where the left and vertical maps are natural inclusions. Let $s : G_k \to G_{X'}$ be a group-theoretic section of the natural projection $G_{X'} \twoheadrightarrow G_k$. Then s induces naturally a group-theoretic section $s' : G_k \to G_E$ of the natural projection $G_E \twoheadrightarrow G_k$. Moreover, one observes easily that the section s arises from a rational point $x' \in X'(k)$ if and only if the section s' arises from a rational point $x \in E(k)$. The section $s' : G_k \to G_E$ arises from a rational point $x \in E(k)$ by Corollary 4.8. $\square$

Remark 5.6. Recently it was proven by Mazur and Rubin (cf. [Mazur-Rubin]) that over any number field k there exist elliptic curves over k with trivial Mordell-Weil rank. As a consequence, Lemma 5.5 implies that for any curve X over k there exists a finite morphism $f : X' \to X$ of degree $\deg(f) \leq 2$ such that the BGASC holds true for X', under the assumption that the Shafarevich-Tate groups of elliptic curves over k are finite.

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